Worksheet #5 solution

1) a) $\sum_{k=3}^{+\infty} \frac{1}{5^k} = \left(\sum_{k=0}^{+\infty} \frac{1}{5^k}\right) - 1 - \frac{1}{5} - \frac{1}{25} = \frac{5}{4} - \frac{31}{25} = \frac{1}{100}$

b) $\sum_{k=50}^{+\infty} \left(\frac{2}{\pi}\right)^k = \left(\sum_{k=0}^{+\infty} \left(\frac{2}{\pi}\right)^k\right) - \left(\sum_{k=0}^{49} \left(\frac{2}{\pi}\right)^k\right) = \frac{1}{1-\left(\frac{2}{\pi}\right)} - \frac{1-\left(\frac{2}{\pi}\right)^{50}}{1-\left(\frac{2}{\pi}\right)} = \frac{\left(\frac{2}{\pi}\right)^{50}}{1-\left(\frac{2}{\pi}\right)}$

c) $\sum_{k=0}^{+\infty} b_k = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{9} + \frac{1}{8} + \frac{1}{27} + \cdots = \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}} = 2 + \frac{3}{2} = \frac{7}{2}$

3) a) By ratio test $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1/(n-1)!}{1/(n-2)!}\right| = \left|\frac{(n-2)!}{(n-1)!}\right| = \frac{1}{n-1}$
then $\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, hence the series converges

b) Taking limits $\lim_{n\to+\infty} a_{n+1} = \frac{\lim_{n\to+\infty} a_n}{2} + \frac{1}{\lim_{n\to+\infty} a_n} \Rightarrow L = \frac{L}{2} + \frac{1}{L} \Rightarrow L^2 = 2$

So L can not be 0. By the n-term test, the series diverges.

c) By root test $\sqrt[n]{|n^{-n}|} = \frac{1}{n}$, then $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 0$, hence the series converges

d) By the n-term test $\lim_{n\to+\infty} \frac{n+1}{\sqrt{n^2+3} - 2} = \lim_{n\to+\infty} \frac{1 + 1/n}{\sqrt{1+\frac{3}{n^2}} - 1/n} = \frac{1+0}{\sqrt{1+0} - 0} = 1 \neq 0$
Hence the series diverges

e) Observe that $\frac{(-1)^{2n}}{3(2n)} + \frac{(-1)^{2n+1}}{3(2n+1)} = \frac{1}{6n(n+1)} < \frac{1}{6n^2}$

then by comparison test the series converges

4) 1) We need that $|e^c| < 1 \Rightarrow e^c < 1$ ($e^c > 0$ always)
then this happens for all $c < 0$

2) Again, we need $|\sqrt{7-x^2}| < 1 \Rightarrow 0 \leq \sqrt{7-x^2} < 1 \Rightarrow 0 \leq 7 - x^2 < 1$
$\Rightarrow 6 < x^2 \leq 7 \Rightarrow \sqrt{6} < x \leq \sqrt{7}$ or $-\sqrt{7} < x \leq -\sqrt{6}$

$x \in (-\sqrt{7}, -\sqrt{6}] \cup [\sqrt{6}, \sqrt{7})$